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A Remark on One Family of Iterative Formulas

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Abstract. In this paper we obtain one family of iterative formulas of the second order for finding zeros of a given function $F(x)$.

In this paper, starting from the Newton's iterative formula

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)},\ k = 0, 1, 2, \dots, \tag{1}$$

which we write in the form

$$x_{k+1} = x_k \left(1 - \frac{F(x_k)}{x_k F'(x_k)}\right),\ k = 0, 1, 2, \dots, \tag{2}$$

we obtain a family of iterative formulas of the second order.

If, instead of

$$1 - \frac{F(x_k)}{F'(x_k)},$$

we put

$$\left(1 - \frac{F(x_k)}{s x_k F'(x_k)}\right)^s,$$

in formula (2), where $s \neq 0$ is a real parameter, we get an iterative formula

$$x_{k+1} = x_k \left(1 - \frac{F(x_k)}{s x_k F'(x_k)}\right)^s,\ k = 0, 1, 2 \dots \tag{3}$$

The expression (3) represents one family of iterative formulas, i.e. one iterative method, of the second order.

If we now consider different real values for the parameter s, we obtain particular iterative formulas from (3).

For $s = 1$, (3) reduces to Newton's method (2).

For finding zeros of the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{4}$$

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of the degree n, we take $s = \frac{1}{n}$ if n is odd. In this case, the formula (3) reduces to

$$x_{k+1} = x_k \left(1 - \frac{nP(x_k)}{x_k P'(x_k)}\right)^{\frac{1}{n}}, \ k = 0, 1, 2, \dots \tag{5}$$

If n is even, then we take $s = \frac{1}{n-1}$. In that case, (3) becomes

$$x_{k+1} = x_k \left(1 - \frac{(n-1)P(x_k)}{x_k P'(x_k)}\right)^{\frac{1}{n-1}}, \ k = 0, 1, 2, \dots \tag{6}$$

The location of the zeros of the polynomial (4) in the complex plane, depending on its coefficients a_k, where $k = 0, 1, 2, n$, were studied by many authors (see, e.g. [**1**]). Here we cite two results due to Cauchy [**1**, pp. 122–123] and a result due to P. Montel [**2**] which are, respectively, as follows:

(R_1) All the zeros of the polynomial (4) lie in the circle

$$|x| \le r, \tag{7}$$

where r is a positive root of the equation

$$P_1(x) = |a_n|x^n - |a_{n-1}|x^{n-1} - \cdots - |a_1|x - |a_0| = 0. \tag{8}$$

(R_2) All the zeros of the polynomial (4) lie in the region

$$|x| < 1 + A, \tag{9}$$

where

$$A = \max \left| \frac{a_k}{a_n} \right|, \ k = 0, 1, 2, \dots, n-1. \tag{10}$$

(R_3) All the zeros of the polynomial (4) lie in the region

$$|x| < 2M, \tag{11}$$

where

$$M = \max \left| \frac{a_{n-k}}{a_n} \right|, \ k = 1, 2, \dots, n. \tag{12}$$

For determining the upper bound for the moduli of zeros of the polynomial (4), we can use the formula

$$x_{k+1} = x_k \left(1 - \frac{nP_1(x_k)}{x_k P_1'(x_k)}\right)^{\frac{1}{n}}, \ k = 0, 1, 2, \dots \tag{13}$$

The method (13), for $x_0 > 1 + A$, converges monotonically to r.

Example. We determine the upper bound for the moduli of zeros of the polynomial

$$P(x) = x^5 - 5x + 22$$

using the formula (13), where

$$P_1(x) = x^5 - 5x - 22,$$

for $x_0 > 1 + A$.

Taking $x_0 = 30$, we obtain

Newton's method

$$x_{k+1} = x_k - \frac{P_1(x_n)}{P_1'(x_k)}$$
$$x_0 = 30$$
$$x_1 = 24.00003506$$
$$\vdots$$
$$x_5 = 7.866439195$$
$$\vdots$$
$$x_{10} = 2.675267821$$
$$\vdots$$
$$x_{14} = 2.000013231$$
$$x_{15} = 2.000000000$$

Method using formula (13)

$$x_{k+1} = x_k\left(1 - \frac{5P_1(x_k)}{x_k P_1'(x_k)}\right)^{\frac{1}{5}}$$
$$x_0 = 30$$
$$x_1 = 2.69437266$$
$$x_2 = 2.017339895$$
$$x_3 = 2.000019658$$
$$x_4 = 2.000000000$$

References

[1] M. Marden, *Geometry of Polynomials,* Amer. Math. Soc. Providence, R.I. 2005.

[2] P. Montel, *Sur quelques limites pour les modules des zéros des polynômes,* Comment. Math. Helv., Vol. 7, 1934–35, 178–200.

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